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An Efficient Inexact NMPC Scheme with Stability and Feasibility Guarantees

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Abstract: In this paper, an inexact nonlinear model predictive control scheme with reduced computational complexity is proposed. The presented approach exploits fixed sensitivity information precomputed offline at a reference value. This allows one to avoid the online computational effort resulting from the propagation of sensitivities and possibly the corresponding condensing routine when solving the optimal control problem with a sequential quadratic programming method. By performing a numerical simulation of the nonlinear dynamics online, feasibility of the closed-loop trajectories can be preserved in contrast to linear model predictive control schemes. Nominal stability guarantees of the approach are derived and the effectiveness of the scheme is demonstrated on a non-trivial example.

Keywords: nonlinear model predictive control, numerical methods, Lyapunov stability, embedded optimization.

1. INTRODUCTION

Schemes based on Nonlinear Model Predictive Control (NMPC) can be a useful tool to control a large class of systems (Qin and Badgwell, 2000). Due to their inherent flexibility, nonlinear dynamics, constraints and objectives can be tackled directly allowing one to naturally translate design requirements into mathematical statements without the need for ad-hoc reformulations. Moreover, extensive results exist in the literature that guarantee nominal and robust stability under reasonable assumptions (Mayne et al., 2000). Due to the high computational burden associated with the solution of the nonlinear and in general nonconvex Optimal Control Problems (OCP), NMPC has been historically employed mainly in the chemical industry, where the sampling times are generally long enough (García et al., 1989; Qin and Badgwell, 2003).

As more and more efficient numerical methods are being developed and as the computational power available on embedded systems increases, NMPC is becoming a viable way for a broader spectrum of systems. Promising results have been reported for applications in automotive (Frasch et al., 2013), renewable energy (Ferreau et al., 2011) and robotics (Diehl et al., 2006), where sampling times in the millisecond time-scale need to be met.

In this work, motivated by these considerations, an approximate scheme with reduced computational complexity is analyzed and its stability properties are investigated. In the setup considered, the optimal control problem is solved with the direct multiple shooting method using a Sequential Quadratic Programming (SQP) scheme (Bock and Plitt, 1984). In contrast with exact schemes, fixed sensitivities are used in the convex Quadratic Problems (QP) arising in each iteration. In this way, the subproblems can be pre-condensed offline and a cheap online update of the nonlinear constraints is performed in order to recover a feasible, but suboptimal solution as proposed in (Bock et al., 2007). In this last work, the pre-condensed QP is used as a building block of a more articulated scheme, the so called multilevel real-time iteration algorithm.

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1.1 Contribution and Outline

In this paper, the inexact NMPC scheme introduced in (Bock et al., 2007) is regarded as a standalone algorithm and its stability properties are investigated. In particular, a sensitivity analysis of the suboptimal solution is derived for the inexact scheme. It is shown that the cost of the optimization problem can be used as a Lyapunov function in a neighbourhood of the reference equilibrium, such that stability can be guaranteed. With respect to other inexact approaches based on a fixed linearization of nonlinear constraints (De Nicolao et al., 2000), the proposed scheme has the advantage of guaranteeing recursive feasibility of the suboptimal solution.

The theoretical statements are then illustrated on a non-trivial numerical example. The inexact scheme is compared with a second approach that uses instead exact derivatives. Despite the potentially large suboptimality of the inexact scheme, it is shown that the condition necessary to obtain stability guarantees hold in a non-negligible neighbourhood of the reference.

The paper is organized as follows: Section 2 briefly summarizes the main ideas and concepts that will be used to derive the stability proof. The inexact SQP scheme is introduced and a standard nominal stability argument for NMPC is recalled. In Section 3 the main results are presented. The sensitivity of the cost associated with the suboptimal trajectories with respect to the initial condition is derived and used to build a Lyapunov function. Finally, Section 4 contains an illustrative example in which the inexact scheme is applied to a nonlinear dynamical system in order to validate the theoretical results.

2. PRELIMINARIES

In the following, the optimal control problem formulation and the inexact scheme are introduced. Moreover, a standard nominal stability argument for NMPC is recalled as the stability proof for the inexact scheme will be built on it.

2.1 Optimal Control Problem Statement

Consider the following finite-horizon discrete-time optimal control problem:

$$\begin{aligned} V^*(\bar{x}_0) = \min_{\substack{x_0,\cdots,x_N \\ u_0,\cdots,u_{N-1}}} \quad & \frac{1}{2}\sum_{i=0}^{N-1}\left(x_i^T Q x_i + u_i^T R u_i\right) \\ \text{s.t.} \quad & x_0 - \bar{x}_0 = 0 \\ & x_{i+1} = f(x_i, u_i), \quad i \in \mathcal{I} \\ & x_N = 0, \end{aligned} \tag{1}$$

with optimization variables $x_i \in \mathbb{R}^{n_x}$ and $u_i \in \mathbb{R}^{n_u}$, positive definite cost matrices $Q \in \mathbb{S}_{++}^{n_x}$ and $R \in \mathbb{S}_{++}^{n_u}$ and $\mathcal{I} := \{0, \cdots, N-1\}$. Without loss of generality, the origin is considered as the reference equilibrium, i.e $f(0,0) = 0$. Such a formulation can be obtained from a continuous-time problem by applying the direct multiple shooting method (Bock and Plitt, 1984). In this work, an OCP formulation with a zero terminal constraint and without inequality constraints is considered in order to simplify the derivation of stability guarantees for the inexact scheme. Similar arguments could be used to extend the proof to a more general OCP formulation without zero terminal constraints and with inequality constraints.

Problem (1) can be reliably and efficiently solved online in an embedded setting with state-of-the-art solvers for NMPC (Diehl et al., 2009). While other approaches exist, a possible way of doing so is by exploiting an SQP scheme that solves a series of convex Quadratic Programs (QP) that locally approximate the original problem. In the next section, the main idea behind such a method is briefly recalled and the modifications that lead to the proposed inexact approach are introduced.

Throughout this paper the following assumption will be made.

Assumption 1. There exists a local minimizer that satisfies Linear Independence Constraint Qualification (LICQ) and Second Order Sufficient Conditions (SOSC) of optimality (Nocedal and Wright, 2006) for the OCP in (1) for any $\bar{x}_0$ in a non-empty neighbourhood Ω of the origin.

2.2 The Inexact Scheme

If the Nonlinear Program (NLP) in (1) is solved with a Gauss-Newton SQP scheme (Bock, 1983), the resulting subproblem at iteration k takes the form

$$\begin{aligned} \min_{\substack{x_0,\cdots,x_N \\ u_0,\cdots,u_{N-1}}} \quad & \frac{1}{2}\sum_{i=0}^{N-1}\left(x_i^T Q x_i + u_i^T R u_i\right) \\ \text{s.t.} \quad & x_0 - \bar{x}_0 = 0 \\ & x_{i+1} = A_i^k x_i + B_i^k u_i + c_i^k, \quad i \in \mathcal{I} \\ & x_N = 0, \end{aligned} \tag{2}$$

where

$$c_i^k = f(x_i^k, u_i^k) - A_i^k x_i^k - B_i^k u_i^k,$$

with dynamics linearized around the current estimate of the solution $x^k := (x_0^k, \ldots, x_N^k)$ and $u^k := (u_0^k, \ldots, u_{N-1}^k)$:

$$A_i^k = \frac{\partial f}{\partial x}(x_i^k, u_i^k), \qquad B_i^k = \frac{\partial f}{\partial u}(x_i^k, u_i^k).$$

If the solution of the above QP is taken as next solution guess $\left(x^{k+1}, u^{k+1}\right)$, it can be shown that such a scheme, together with an appropriate globalization strategy (Han, 1977), is guaranteed to recover a local minimum of (1).

A possible way of solving the subproblems in (2) is by eliminating the state variables x resulting in a smaller and dense problem that can be efficiently solved by state-of-the-art QP solvers (Ferreau et al., 2014). However, the computational burden introduced by the so called condensing (Bock, 1983) can be rather high, especially for problems with a long prediction horizon N or a large number of states (Vukov et al., 2013). A second source of computational effort, regardless of the QP formulation used, is the fact that the matrices A_i^k and B_i^k need to be computed at every iteration. Numerical results that take into account the recent advances in both integration and condensing routines show that the latter can be rather computationally expensive (Quirynen et al., 2013).

The main motivation behind an inexact scheme that exploits fixed derivatives is to avoid the computational burden associated with condensing and sensitivity generation by using the Jacobians evaluated at the reference:

$$A_i^k = A = \frac{\partial f}{\partial x}(0,0), \qquad B_i^k = B = \frac{\partial f}{\partial u}(0,0).$$

A convergence proof of such an SQP scheme to a feasible, but suboptimal solution can be found in (Bock et al., 2007). Using fixed derivatives will allow one to avoid sensitivity generation and condensing routines while solving the optimal control problem online, hence reducing the cost of the overall scheme to the one associated with a pre-condensed QP solve and a numerical simulation of the dynamics needed to update the terms c_i^k in (2).

2.3 Standard Nominal Stability Argument

Classical nominal and robust stability results hold for the system controlled using the optimal solution of the NLP in (1) in a receding horizon fashion (Scokaert et al., 1997). In the following, a common argument used to derive stability guarantees based on Lyapunov functions is reported.

Consider the following result adapted from (Scokaert et al., 1997):

Theorem 2. Let $F : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ be the function that describes the dynamics of the closed-loop system and let F satisfy a Lipschitz condition in an open neighbourhood of the origin, with $F(0) = 0$. The origin is a locally exponentially stable fixed point of $x_{k+1} = F(x_k)$, if there exist a function $U(x_k)$ and strictly positive constants a, b, c and ρ such that, for all $x_k \in B_\rho$, with $B_\rho := \{x \in \mathbb{R}^n : \|x\| \leq \rho\}$, the following holds:

$$\begin{aligned} &a \|x_k\|^2 \leq U(x_k) \leq b \|x_k\|^2 \\ &U(F(x_k)) - U(x_k) \leq -c \|x_k\|^2 . \end{aligned}$$

It is possible to show that the optimal cost in (1) is a valid Lyapunov function for the system controlled by applying the optimal input in a receding horizon fashion. To this end, let $V^*(x_k)$ denote the optimal cost associated with problem (1) for $\bar{x}_0 = x_k$. Then, due to positive definiteness of Q and R, it immediately follows that there exists a constant $a \geq 0$ such that

$$V^*(x_k) \geq a \|x_k\|_2^2 .$$

Consider the control profile $u^*(x_k)$ obtained by solving problem (1) with $\bar{x}_0 = x_k$. As the optimal solution satisfies $x_N^* = 0$, a feasible solution to problem (1) formulated with $\bar{x}_0 = x_{k+1} = f(x_k, u_0^*(x_k))$ can be obtained by replaying the controls computed at time k:

$$\hat{u} := \left(u_1^*(x_k),\, u_2^*(x_k),\, \cdots,\, u_{N-1}^*(x_k),\, 0\right),$$

which would give rise to a feasible and suboptimal solution of (1) with cost

$$V^*(x_k) - x_k^T Q x_k - u_0^*(x_k)^T R u_0^*(x_k).$$

The optimal cost $V^*(x_{k+1})$ can only be better than this, thus:

$$V^*(x_{k+1}) - V^*(x_k) \leq -x_k^T Q x_k - u_0^*(x_k)^T R u_0^*(x_k),$$

or, equivalently, there exists a constant c such that

$$V^*(x_{k+1}) - V^*(x_k) \leq -c \|x_k\|_2^2 , \quad \forall x_k \in B_\rho.$$

Finally, Assumption 1 can be used to derive an appropriate constant $b > 0$. The proof is not reported here for brevity and the interested reader is referred to (Scokaert et al., 1997) for a detailed discussion.

3. STABILITY OF THE INEXACT NMPC SCHEME

In the following, a sensitivity analysis of the solution recovered by the inexact scheme will be derived. In order to do so, regard the first order optimality conditions (Nocedal and Wright, 2006) for the QP subproblem (2) at iteration k, with fixed derivatives A and B:

$$\begin{aligned} &x_0 - \bar{x}_0 &&= 0 \\ &Qx_0 + A^T\lambda_1 - \lambda_0 &&= 0 \\ &Ru_0 + B^T\lambda_1 &&= 0 \\ &x_1 - f(x_0^k, u_0^k) - A(x_0 - x_0^k) - B(u_0 - u_0^k) &&= 0 \\ &\vdots && \vdots \\ &x_N &&= 0 \\ &-\lambda_N - \lambda_t &&= 0. \end{aligned} \tag{3}$$

where $\lambda_0, \cdots, \lambda_N, \lambda_t$ are the Lagrange multipliers associated with the equality constraints. When convergence is achieved $x_i^{k+1} = x_i^k := \tilde{x}_i$, $u_i^{k+1} = u_i^k := \tilde{u}_i$ and $\lambda_i^{k+1} = \lambda_i^k := \tilde{\lambda}_i$ and the following holds:

$$\begin{aligned} &\tilde{x}_0 - \bar{x}_0 &&= 0 \\ &Q\tilde{x}_0 + A^T\tilde{\lambda}_1 - \tilde{\lambda}_0 &&= 0 \\ &R\tilde{u}_0 + B^T\tilde{\lambda}_1 &&= 0 \\ &\tilde{x}_1 - f(\tilde{x}_0, \tilde{u}_0) &&= 0 \\ &\vdots && \vdots \\ &\tilde{x}_N &&= 0 \\ &-\tilde{\lambda}_N - \tilde{\lambda}_t &&= 0. \end{aligned} \tag{4}$$

The solution recovered by the inexact scheme is feasible, as the equality constraints appear unchanged in the nonlinear root-finding problem (4). The approximation introduced by fixing the sensitivities affects instead the optimality of the solution. For this reason, when analyzing the stability of the closed-loop scheme, the question naturally arises of how this approximation degrades the properties of the optimal cost which is used as a Lyapunov function in exact schemes.

The nominal stability proof will be based on a sensitivity analysis of the solution to the nonlinear root-finding problem (4) with respect to the initial value $\bar{x}_0$ at the origin. These considerations will be then exploited to prove that, in a neighbourhood of the origin, the standard Lyapunov stability arguments must hold for the inexact scheme as well.

3.1 Sensitivity of the Suboptimal Solution

First, a result on the sensitivity of both optimal and suboptimal solutions with respect to $\bar{x}_0$ will be derived. Let $w^* = (x^*, u^*, \lambda^*)$ be the optimal solution associated with the original NLP (1), hence satisfying the following optimality conditions:

$$\begin{aligned} &x_0^* - \bar{x}_0 &&= 0 \\ &Qx_0^* + \nabla_x f(x_0^*, u_0^*)\lambda_1^* - \lambda_0^* &&= 0 \\ &Ru_0^* + \nabla_u f(x_0^*, u_0^*)\lambda_1^* &&= 0 \\ &x_1^* - f(\bar{x}_0, u_0^*) &&= 0 \\ &\vdots && \vdots \\ &x_N^* &&= 0 \\ &-\lambda_N^* - \lambda_t^* &&= 0, \end{aligned} \tag{5}$$

where $\nabla_x f$ and $\nabla_u f$ denote the transpose of the Jacobian of f with respect to x and u respectively. Equations (4) and (5) will be respectively referred to in the compact forms

$$\tilde{q}(\bar{x}_0, \tilde{w}) = 0 \quad \text{and} \quad q^*(\bar{x}_0, w^*) = 0,$$

where $\tilde{w} = (\tilde{x}, \tilde{u}, \tilde{\lambda})$ is the vector containing the concatenated suboptimal primal and dual solution. The following result holds:

Lemma 3. The magnitude of the deviation of the suboptimal solution $\tilde{w}$ from the optimal one w^* is of second order in the magnitude of the norm of the initial condition $\bar{x}_0$:

$$\|\tilde{w}(\bar{x}_0) - w^*(\bar{x}_0)\| = \mathcal{O}\left(\|\bar{x}_0\|^2\right).$$

Proof. By exploiting the implicit function theorem, it is possible to express the sensitivity of optimal and suboptimal solutions with respect to the initial condition $\bar{x}_0$ around the origin respectively as

$$\frac{\partial w^*}{\partial \bar{x}_0} = -\frac{\partial q^*}{\partial w^*}^{-1}\frac{\partial q^*}{\partial \bar{x}_0} \quad \text{and} \quad \frac{\partial \tilde{w}}{\partial \bar{x}_0} = -\frac{\partial \tilde{q}}{\partial \tilde{w}}^{-1}\frac{\partial \tilde{q}}{\partial \bar{x}_0},$$

where invertibility of $\frac{\partial q^*}{\partial w^*}$ and $\frac{\partial \tilde{q}}{\partial \tilde{w}}$ is guaranteed in a neighbourhood of the origin due to Assumption 1. It is possible to see that the derivative matrices coincide at the origin, i.e

$$\frac{\partial \tilde{q}}{\partial \tilde{w}}(0,0) = \frac{\partial q^*}{\partial w^*}(0,0) \quad \text{and} \quad \frac{\partial \tilde{q}}{\partial \bar{x}_0}(0,0) = \frac{\partial q^*}{\partial \bar{x}_0}(0,0)$$

due to the fact that

$$A = \frac{\partial f}{\partial x}(0,0) \quad \text{and} \quad B = \frac{\partial f}{\partial u}(0,0).$$

This implies that both the constant and linear term of the Taylor series of $\tilde{w}(\bar{x}_0)$ and $w^*(\bar{x}_0)$ coincide at the origin $\bar{x}_0 = 0$. ■

3.2 Suboptimality of the Inexact Scheme

In the following the OCP in (1) will be regarded in the compact form

$$\begin{aligned} \min_{z} \quad & J(z) \\ \text{s.t.} \quad & G(\bar{x}_0, z) = 0, \end{aligned} \tag{6}$$

where $z := (x, u)$ has been introduced to refer to the primal solution in compact form. The result from Lemma 3 can be used to quantify the suboptimality of $\tilde{w}(\bar{x}_0)$ as a function of $\|\bar{x}_0\|$. In particular, the following holds:

Lemma 4. The deviation of the cost $J(\tilde{z})$ associated with the suboptimal solution from the optimal one $J(z^*)$ is of fourth order in $\|\bar{x}_0\|$:

$$J\left(\tilde{z}(\bar{x}_0)\right) - J\left(z^*(\bar{x}_0)\right) = \mathcal{O}\left(\|\bar{x}_0\|^4\right).$$

Proof. The proof exploits the fact that $\tilde{z}$ is a feasible solution, hence $G(\bar{x}_0, \tilde{z}) = 0$. Thus, objective and Lagrangian coincide $J(\tilde{z}) = \mathcal{L}(\tilde{z}, \lambda)$, where:

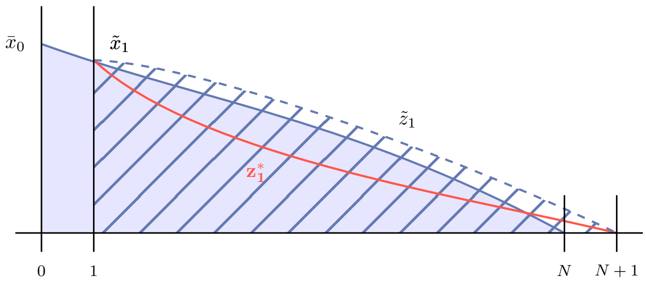

Fig. 1. The sketch describes the procedure followed in the proof of Theorem 5. The cost associated with the suboptimal trajectory starting at $\bar{x}_0$ is first compared with the one associated with the optimal trajectory starting at $\tilde{x}_1$ - in red. Then, due to Lemma 4, the cost increases with the fourth power of $\|\tilde{x}_1\|$ going from z_1^* to $\tilde{z}_1$.

$$\mathcal{L}(z, \lambda) := J(z) - \lambda^T G(z).$$

The Taylor series of the Lagrangian around the optimal solution (z^*, λ^*) reads

$$\mathcal{L}(z, \lambda^*) = \mathcal{L}(z^*, \lambda^*) + \nabla_z \mathcal{L}(z^*, \lambda^*)^T (z - z^*) + \mathcal{O}\left(\|z - z^*\|^2\right).$$

Using the fact that z^* is an optimal solution, i.e. $\nabla_z \mathcal{L}(z^*, \lambda^*) = 0$, the following is obtained for $z = \tilde{z}$:

$$\begin{aligned} J(\tilde{z}) = \mathcal{L}(\tilde{z}, \lambda^*) \quad &= \mathcal{L}(z^*, \lambda^*) + \mathcal{O}\left(\|(\tilde{z} - z^*)\|^2\right) \\ &= J(z^*) + \mathcal{O}\left(\|(\tilde{z} - z^*)\|^2\right). \end{aligned}$$

Together with the fact that $\|\tilde{w}(\bar{x}_0) - w^*(\bar{x}_0)\| = \mathcal{O}\left(\|\bar{x}_0\|^2\right)$, this implies that the suboptimality grows with the fourth power of the norm of the initial condition:

$$\|J\left(\tilde{z}(\bar{x}_0)\right) - J\left(z^*(\bar{x}_0)\right)\| = \mathcal{O}\left(\|\bar{x}_0\|^4\right). \quad ■$$

3.3 A Lyapunov Function for the Inexact Scheme

The observations made in the previous lemmata and theorems will be now exploited in order to build a Lyapunov function for the system controlled by applying the suboptimal solution in a receding horizon fashion. The following theorem states the main stability result for the inexact scheme:

Theorem 5. The origin is a locally exponentially stable equilibrium for the closed-loop system obtained by applying the suboptimal control input $\tilde{u}_0(\bar{x}_0)$.

Proof. In order to prove the above result, it is possible to rely on Theorem 2. It will be shown that the cost

$$\tilde{V}(\bar{x}_0) := J\left(\tilde{z}(\bar{x}_0)\right)$$

is a valid Lyapunov function. For a given initial condition $\bar{x}_0$, the following holds:

$$\tilde{V}(\bar{x}_0) \geq \frac{1}{2}\bar{x}_0^T Q \bar{x}_0 + V^*(\tilde{x}_1)$$

and, due to Lemma 4,

$$\tilde{V}(\bar{x}_0) \geq \bar{x}_0^T Q \bar{x}_0 + \tilde{V}(\tilde{x}_1) + \mathcal{O}\left(\|\tilde{x}_1\|^4\right).$$

This last inequality shows that $\tilde{V}(\cdot)$ decreases for $\|\tilde{x}_1\|$ small enough. The trajectories taken into account to build the proof are illustrated in Figure 1. As the primal solution grows linearly with $\bar{x}_0$, i.e. $\tilde{x}_1 = \mathcal{O}(\|\bar{x}_0\|)$, the following holds:

$$\tilde{V}(\bar{x}_0) \geq \bar{x}_0^T Q \bar{x}_0 + \tilde{V}(\tilde{x}_1) + \mathcal{O}\left(\|\bar{x}_0\|^4\right).$$

Hence, there exist strictly positive constants ρ and c such that

$$\tilde{V}(\tilde{x}_1) - \tilde{V}(\bar{x}_0) \leq -c\,\|\bar{x}_0\|^2, \quad \forall \bar{x}_0 \in B_\rho.$$

Moreover, noting that

$$J(\tilde{z}(\bar{x}_0)) = \tilde{z}(\bar{x}_0)^T H \tilde{z}(\bar{x}_0),$$

where H is the Hessian of the cost J, together with the well-posedness of the problem introduced in Assumption 1, makes it possible to define a constant $b > 0$ such that

$$\tilde{V}(\bar{x}_0) \leq b\,\|\bar{x}_0\|^2.$$

Finally, a positive constant a such that $\tilde{V}(\bar{x}_0) \geq a\,\|\bar{x}_0\|^2$ can be derived as previously done for Theorem 2.

■

4. ILLUSTRATIVE EXAMPLE

In this section, the inexact scheme will be applied to a non-trivial example and it will be shown that the conditions necessary for stability hold in a non-negligible neighbourhood of the origin, as shown in Theorem 5.

An OCP of the form in (1) is considered, where $f(\cdot)$ represents discretized dynamics obtained by applying the explicit Runge-Kutta scheme of order 4 with fixed step-size $h = 0.1$ to the following ordinary differential equation:

$$\dot{x} = f_c(x, u) := \begin{bmatrix} x_1^3 + (1 + x_2)u_1 \\ x_2^3 + x_1 + u_2 \end{bmatrix}.$$

A control horizon $T = 1$ is used and the trajectories are discretized using $N = 10$ shooting nodes. The cost matrices have been chosen to be equal to the identity matrix $Q = R = \mathbf{I}_2$.

The full-step Gauss-Newton algorithm will be used for the comparison and both the exact and inexact SQP-type algorithms will be iterated until either convergence or failure. Two possible causes of failure are taken into account: either the algorithm has not converged after a maximum number of iterations $\tau_{\max} = 100$ or an infeasible QP has arisen.

The state space region $\mathcal{X} = \{-1.2 \leq x_1 \leq 1.2, -1.2 \leq x_2 \leq 1.2\}$ is discretized with an equally spaced grid and

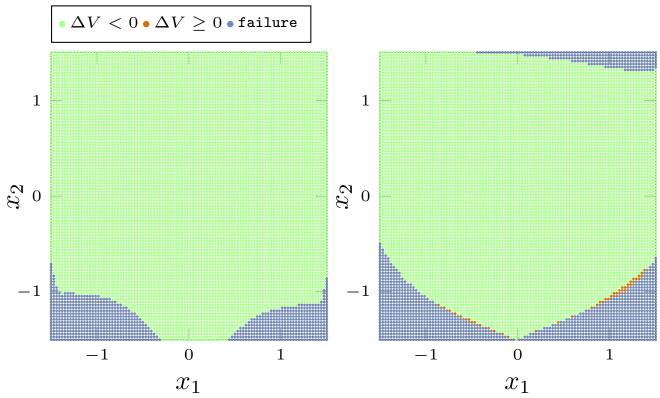

Fig. 2. ΔV for exact (left) and inexact (right) scheme. Decreasing cost $\Delta V < 0$ in green, non-decreasing cost $\Delta V \geq 0$ in red, maximum number of iterations reached or infeasible SQP step in blue. For the exact scheme, the cost is guaranteed to decrease by construction. For the inexact one, the cost can be non-decreasing due to the approximation introduced. However, $\Delta V < 0$ holds in a non-negligible region around the origin.

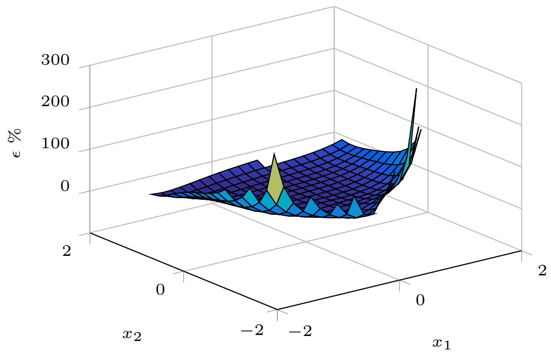

Fig. 3. Relative suboptimality $\epsilon_{\%} = \frac{\tilde{V} - V^*}{V^*} \cdot 100$. The inexact scheme gives rise to largely suboptimal policies in certain regions of the state space, however, as shown in Figure 2 stability is guaranteed in a non-negligible region of the state space.

the OCP is solved with both methods. For any initial condition $\bar{x}_0$ and input computed u_0, let ΔV be the cost difference defined as

$$\Delta V := V\left(f(\bar{x}_0, u_0)\right) - V(\bar{x}_0).$$

Figure 2 shows the regions where $\Delta V < 0$, $\Delta V \geq 0$ or a failure is encountered, comparing the results obtained with the two schemes. In particular, it is shown that stability guarantees can be obtained for the inexact scheme in a rather large neighbourhood of the origin.

The relative suboptimality $\epsilon_{\%} = \frac{\tilde{V} - V^*}{V^*} \cdot 100$ of the trajectories obtained with the inexact scheme is plotted in Figure 3 for different initial conditions. The scheme can become largely suboptimal for points sufficiently distant from the origin, where the fixed sensitivities might not be good approximations of the exact ones. However, closed-loop feasibility is always guaranteed and stability can be guaranteed in a non-negligible region as illustrated in Figure 2.

Finally, in Figure 4, points on a line that passes through the origin parametrized with the scalar coordinate x_d are considered $[x_1\ x_2] = x_d\,[1\ 1.71]$ and the absolute suboptimality $\tilde{V} - V^*$ is compared with $\frac{1}{2}\bar{x}_0^T Q \bar{x}_0$. Due to Lemma 4, the suboptimality is of fourth order in $\|\bar{x}_0\|$, hence the inequality $\tilde{V} - V^* < \frac{1}{2}\bar{x}_0^T Q \bar{x}_0$ holds in a neighbourhood of the origin.

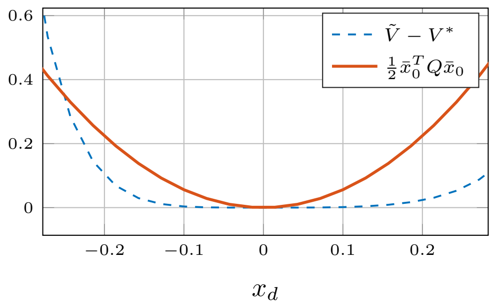

Fig. 4. Absolute suboptimality as a function of the directional coordinate x_d. $\tilde{V} - V^*$ is compared with $\frac{1}{2}\bar{x}_0^T Q \bar{x}_0$. Due to Lemma 4, the suboptimality is dominated by the quadratic cost associated with stage 0 of the OCP.

5. CONCLUSIONS AND OUTLOOK

An efficient inexact scheme based on the ideas presented in (Bock et al., 2007) has been introduced and nominal stability guarantees have been derived. The scheme exploits fixed sensitivity information obtained by evaluating the derivatives at the reference and allows to avoid computationally intensive condensing and sensitivity generation routines. It has been shown that the cost function in problem (1), commonly used for the stability proof of exact schemes, can be used as a Lyapunov function for the inexact approach as well. An illustrative example is given in which it is shown how the necessary conditions for the cost to be usable as a Lyapunov function hold in a rather large region around the origin.

Future work will include an efficient implementation of the algorithm and extensive benchmarking. Moreover the stability results could be extended in order to take into account an approximate scheme where a single QP is solved per iteration as proposed in (Diehl et al., 2007).

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